

A NOTE ON THE CODIMENSION OF THE LINEAR SECTION OF THE LAGRANGIAN-GRASSMANNIAN $L(6, 12)$

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ABSTRACT. Consider a $2n$ -dimensional symplectic vector space E over an arbitrary field \mathbb{F} . Given a contraction map $f : \wedge^n E \rightarrow \wedge^{n-2} E$ such that the Lagrangian–Grassmannian $L(n, 2n) = G(n, 2n) \cap \mathbb{P}(\ker f)$, where $\wedge^r E$ denotes the r -th exterior power of E and $\mathbb{P}(\ker f)$ is the projectivization of $\ker f$. In this paper, for a symplectic vector space E of dimension $n = 6$, we prove that the surjectivity of the contraction map $f : \wedge^6 E \rightarrow \wedge^4 E$ depends on the characteristic of the base field and we calculate the codimension of the linear section $\mathbb{P}(\ker f) \subseteq \mathbb{P}(\wedge^6 E)$ for any characteristic.

1. INTRODUCTION

Let E be a $2n$ -dimensional symplectic vector space over an arbitrary field \mathbb{F} with symplectic form $\langle \cdot, \cdot \rangle$. Consider the contraction map $f : \wedge^n E \rightarrow \wedge^{n-2} E$ given by

$$(1.1) \quad f(w_1 \wedge \cdots \wedge w_n) = \sum_{1 \leq s < t \leq n} \langle w_s, w_t \rangle w_1 \wedge \cdots \wedge \widehat{w}_s \wedge \cdots \wedge \widehat{w}_t \wedge \cdots \wedge w_n,$$

where \widehat{w} means that the corresponding term is omitted. Our main result shows that, in general, the map f is not surjective. Since, by [2] the Lagrangian–Grassmannian variety $L(n, 2n)$ is cut out by the projectivization $\mathbb{P}(\ker f)$ of the kernel of f , it follows that the codimension of $L(n, 2n)$ in its Plücker embedding is not $C_n^{2n} - C_n^{2n-2}$, where C_n^m denotes the binomial coefficient. Specifically, we prove that for $n = 6$, and a field of characteristic 3, the contraction map $f : \wedge^6 E \rightarrow \wedge^4 E$ given by (1.1) is not surjective. To prove this, we use a combinatorial description in Lemma 1 of the set of indices that label the Plücker linear relations that is then used to describe the linear section $\mathbb{P}(\ker f)$ that cuts out the Lagrangian–Grassmannian $L(6, 12)$ in the Grassmannian variety $G(6, 12)$ in any characteristic. As a consequence we show that the codimension of $L(6, 12)$ in its Plücker embedding depends of the characteristic of the base field.

The paper is organized as follows. In Section 2 we recall some results of the contraction map (1.1) and the Lagrangian–Grassmannian. In Section 3 we give an explicit example where the contraction map is not surjective and give all the details involved to obtain the linear section $\mathbb{P}(\ker f)$ that defines $L(6, 12)$.

2. PRELIMINARIES

Let E be a $2n$ -dimensional vector space over \mathbb{F} equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Define the set $I(\ell, m) = \{\alpha = (\alpha_1, \dots, \alpha_\ell) : 1 \leq \alpha_1 < \cdots <$

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$\alpha_\ell \leq m\}$, such that $\alpha_i \in \mathbb{N}$, and the support of $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$ as the set $\text{supp}(\alpha) := \{\alpha_1, \dots, \alpha_\ell\}$. Thus, all indices α are ordered sets of ℓ different integers in the set $\{1, 2, \dots, m\}$. In what follows, all indices α are sets with ℓ different elements in the set $\{1, 2, \dots, m\}$, and up to permutation, we may (and do so) think of them in $I(\ell, m)$.

Choose a basis $\{e_1, \dots, e_{2n}\}$ of the symplectic space E such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = 2n - i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $\alpha = (\alpha_1, \dots, \alpha_n) \in I(n, 2n)$ write

$$\begin{aligned} e_\alpha &:= e_{\alpha_1} \wedge \dots \wedge e_{\alpha_n}, \\ e_{\alpha_{st}} &:= e_{\alpha_1} \wedge \dots \wedge \widehat{e}_{\alpha_s} \wedge \dots \wedge \widehat{e}_{\alpha_t} \wedge \dots \wedge e_{\alpha_n}, \\ p_{i\alpha_{st}(2n-i+1)} &:= p_{i\alpha_1 \dots \widehat{\alpha_s} \dots \widehat{\alpha_t} \dots \alpha_n(2n-i+1)}, \end{aligned}$$

where \widehat{e}_{α_k} and $\widehat{\alpha}_k$ means that the corresponding term is omitted. Denote by $\wedge^n E$ the n -th exterior power of E , which is generated by $\{e_\alpha : \alpha \in I(n, 2n)\}$. For $w = \sum_{\alpha \in I(n, 2n)} p_\alpha e_\alpha \in \wedge^n E$, the coefficients p_α are the *Plücker coordinates* of w . In [2, Proposition 6] the kernel of the contraction map f is characterized as follows: For $w = \sum_{\alpha \in I(n, 2n)} p_\alpha e_\alpha \in \wedge^n E$ written in Plücker coordinates, we have that

$$w \in \ker f \iff \sum_{i=1}^n p_{i\alpha_{st}(2n-i+1)} = 0, \text{ for all } \alpha_{st} \in I(n-2, 2n).$$

In [2, Section 3] these linear forms were given the following description: For $\alpha_{st} \in I(n-2, 2n)$ define the linear polynomials

$$\Pi_{\alpha_{st}} := \sum_{i=1}^n c_{i, \alpha_{st}, 2n-i+1} X_{i, \alpha_{st}, 2n-i+1},$$

with

$$c_{i, \alpha_{st}, 2n-i+1} = \begin{cases} 1 & \text{if } |\text{supp}\{i, \alpha_{st}, 2n-i+1\}| = n, \\ 0 & \text{otherwise,} \end{cases}$$

hence $\Pi_{\alpha_{st}}$ are polynomials in the ring $\mathbb{F}[X_\alpha : \alpha \in I(n, 2n)]$. From the following formula in Plücker coordinates

$$(2.1) \quad X_{1 \sqcup 2n} + X_{2 \sqcup (2n-1)} + \dots + X_{n \sqcup (n+1)} = 0,$$

where the symbols \sqcup are to be replaced by elements $\alpha_{st} \in I(n-2, 2n)$, we obtain homogeneous linear equations, that we call a *Plücker linear relations* in k -variables,

$$(2.2) \quad \Pi_{\alpha_{st}} := X_{1, \alpha_{st}, 2n} + X_{2, \alpha_{st}, (2n-1)} + \dots + X_{n, \alpha_{st}, (n+1)} = 0$$

where the term $X_{i, \alpha_{st}, (2n-i+1)}$ does not appear if $|\text{supp}\{i, \alpha_{st}, (2n-i+1)\}| < n$. When this happens we say $\Pi_{\alpha_{st}}$ is a k -plane. For the system of homogeneous linear equations $\Pi_{\alpha_{st}}, \alpha_{st} \in I(n-2, 2n)$, we denote by B its associated matrix. Clearly the matrix B is of order $C_{n-2}^{2n} \times C_n^{2n}$. For example, if $n = 6$ formula (2.2) becomes

$$(2.3) \quad X_{1 \sqcup C} + X_{2 \sqcup B} + X_{3 \sqcup A} + X_{4 \sqcup 9} + X_{5 \sqcup 8} + X_{6 \sqcup 7} = 0,$$

where $A = 10, B = 11, C = 12$.

Recall that a vector subspace W of E is *isotropic* iff for all $x, y \in W$ we have that $\langle x, y \rangle = 0$, and if W is isotropic its dimension is at most n . The *Lagrangian-Grassmannian*

$L(n, 2n)$ is the projective variety given by the isotropic vector subspaces $W \subseteq E$ of maximal dimension n :

$$L(n, 2n) = \{W \in G(n, 2n) : W \text{ is isotropic and } n\text{-dimensional}\},$$

where $G(n, 2n)$ denotes the Grassmannian variety of vector subspaces of dimension n of E . The *Plücker embedding* is the regular map $\rho : G(n, 2n) \rightarrow \mathbb{P}(\wedge^n E)$ given on each $W \in G(n, 2n)$ by choosing a basis w_1, \dots, w_n of W and then mapping the vector subspace $W \in G(n, 2n)$ to the tensor $w_1 \wedge \dots \wedge w_n \in \wedge^n E$. Since choosing a different basis of W changes the tensor $w_1 \wedge \dots \wedge w_n$ by a nonzero scalar, this tensor is a well-defined element in the projective space $\mathbb{P}(\wedge^n E) \simeq \mathbb{P}^{N-1}$, where $N = C_n^{2n} = \dim_{\mathbb{F}}(\wedge^n E)$. Under the Plücker embedding, the Lagrangian-Grassmannian is given by

$$L(n, 2n) = \{w_1 \wedge \dots \wedge w_n \in G(n, 2n) : \langle w_i, w_j \rangle = 0 \text{ for all } 1 \leq i < j \leq n\}.$$

Using the contraction map $f : \wedge^n E \rightarrow \wedge^{n-2} E$ given by (1.1), if $\mathbb{P}(\ker f)$ is the projectivization of $\ker f$, in [2] it is proved that $L(n, 2n) = G(n, 2n) \cap \mathbb{P}(\ker f)$. We call $\mathbb{P}(\ker f)$ the linear section that defines $L(n, 2n)$ in $\mathbb{P}(\wedge^n E)$.

3. NON SURJECTIVITY OF THE CONTRACTION MAP IN $\text{char}(\mathbb{F}) = 3$

The purpose of this section is twofold: First, to provide a description, completely explicit and self-contained of the linear space $\mathbb{P}(\ker f)$ for $n = 6$, for any field \mathbb{F} , and then using this characterization we give an example of the non surjectivity of the contraction map for a field of characteristic 3.

Let $P_1 = (1, C), P_2 = (2, B), P_3 = (3, A), P_4 = (4, 9), P_5 = (5, 8), P_6 = (6, 7)$, where $A = 10, B = 11, C = 12$, as in Section 2, $\Sigma_6 = \{P_1, P_2, \dots, P_6\}$, and $C_2(\Sigma_6)$ the set of all combinations of 6 objects taken 2 at a time. For $1 \leq \alpha_1 < \alpha_2 \leq 12$ such that $\alpha_1 + \alpha_2 \neq 13$, we define the following set

$$\Sigma\{\alpha_1, \alpha_2\} = \{(\alpha_1, \alpha_2, P_i) \in I(4, 12) : i + \alpha_j \neq 13, \alpha_j + 13 - i \neq 13, j = 1, 2\}.$$

Now, for $1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12$ such that $\alpha_i + \alpha_j \neq 13$, define

$$\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in I(4, 12) : \alpha_i + \alpha_j \neq 13 \text{ with } 1 \leq i, j \leq 4\}.$$

Lemma 1. *With the notation above we have a partition of $I(4, 12)$, given by*

$$C_2(\Sigma_6) \cup \left(\bigcup_{\substack{1 \leq \alpha_1 < \alpha_2 \leq 12 \\ \alpha_1 + \alpha_2 \neq 13}} \Sigma\{\alpha_1, \alpha_2\} \right) \cup \left(\bigcup_{\substack{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12 \\ \alpha_i + \alpha_j \neq 13}} \Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \right).$$

Proof. It is enough to show that every element in $I(4, 12)$ is included in one and only one of the three different types of sets on the right hand side of the equality. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in I(4, 12)$. For $\alpha_i + \alpha_j \neq 13$, where $i, j = 1, 2, 3, 4$, it follows that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. If for some $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ we have $\alpha_1 + \alpha_2 \neq 13$, without loss of generality we may assume that $\alpha_3 + \alpha_4 = 13$, and then $\alpha \in \Sigma\{\alpha_1, \alpha_2\}$. Finally, if for some $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 13$, then $\alpha \in C_2(\Sigma_6)$. \square

Remark 1. For each $1 \leq \alpha_1 < \alpha_2 \leq 12$ such that $\alpha_1 + \alpha_2 \neq 13$, we have that $|\Sigma\{\alpha_1, \alpha_2\}| = 4$ and $|\Sigma\{\alpha_1, \alpha_2\} : 1 \leq \alpha_1 < \alpha_2 \leq 12| = 60$. Hence, in the second term of the displayed expression in Lemma 1 there are 240 indexes. Also, for each $1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12$ such that $\alpha_i + \alpha_j \neq 13$, we have that $|\Sigma(\alpha_1, \alpha_2, \alpha_3, \alpha_4)| = 1$, and $|\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} : 1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12| = 240$. Hence, in the third term of the displayed expression in Lemma 1 there are 240 indexes.

We traslate now the combinatorial data of Lemma 1 in terms of the systems of linear equations associated to the contraction map $f : \wedge^6 E \longrightarrow \wedge^4 E$. For each $\alpha_{rs} \in I(4, 12)$ consider the linear equation (2.2). Now, for the part $C_2(\Sigma_6)$ in Lemma 1, writing

$$C_2(\Sigma_6) = \{(P_1, P_2), (P_1, P_3), (P_1, P_4), (P_1, P_5), (P_1, P_6), (P_2, P_3), (P_2, P_4), (P_2, P_5), (P_2, P_6), (P_3, P_4), (P_3, P_5), (P_3, P_6), (P_4, P_5), (P_4, P_6), (P_5, P_6)\}.$$

For the set $C_2(\Sigma_6)$ ordered as above, filling the symbols \sqcup in (2.3) we obtain the system of linear equations

$$(3.1) \quad \begin{aligned} \Pi_{(P_1 P_2)} &:= X_{P_3 P_1 P_2} + X_{P_4 P_1 P_2} + X_{P_5 P_1 P_2} + X_{P_6 P_1 P_2} = 0 \\ \Pi_{(P_1 P_3)} &:= X_{P_2 P_1 P_3} + X_{P_4 P_1 P_3} + X_{P_5 P_1 P_3} + X_{P_6 P_1 P_3} = 0 \\ \Pi_{(P_1 P_4)} &:= X_{P_2 P_1 P_4} + X_{P_3 P_1 P_4} + X_{P_5 P_1 P_4} + X_{P_6 P_1 P_4} = 0 \\ \Pi_{(P_1 P_5)} &:= X_{P_2 P_1 P_5} + X_{P_3 P_1 P_5} + X_{P_4 P_1 P_5} + X_{P_6 P_1 P_5} = 0 \\ \Pi_{(P_1 P_6)} &:= X_{P_2 P_1 P_6} + X_{P_3 P_1 P_6} + X_{P_4 P_1 P_6} + X_{P_5 P_1 P_6} = 0 \\ \Pi_{(P_2 P_3)} &:= X_{P_1 P_2 P_3} + X_{P_4 P_1 P_3} + X_{P_5 P_1 P_3} + X_{P_6 P_1 P_3} = 0 \\ &\vdots \\ \Pi_{(P_5 P_6)} &:= X_{P_1 P_5 P_6} + X_{P_2 P_5 P_6} + X_{P_3 P_5 P_6} + X_{P_4 P_5 P_6} = 0. \end{aligned}$$

where we identify the variable $X_{P_i P_j P_k} = X_{P'_i P'_j P'_k}$ if $\text{supp}\{P_i P_j P_k\} = \text{supp}\{P'_i P'_j P'_k\}$.

Similarly, for the second part in the partition of $I(4, 12)$ in Lemma 1, for the sets $\Sigma\{\alpha_1, \alpha_2\}$, for each $1 \leq \alpha_1 < \alpha_2 \leq 12$, consider the system of four homogeneous linear equations $\Pi_{\alpha_{rs}}$ of (2.3), for each $\alpha_{rs} \in \Sigma\{\alpha_1 \alpha_2\}$, which have the form:

$$(3.2) \quad \begin{aligned} X_1 + X_2 + X_3 &= 0 \\ X_1 + X_4 + X_5 &= 0 \\ X_2 + X_4 + X_6 &= 0 \\ X_3 + X_5 + X_6 &= 0. \end{aligned}$$

and there are 60 such systems of linear equations (3.2). For example, for $\Sigma\{1, 2\} = \{12P_3, 12P_4, 12P_5, 12P_6\}$, setting $A = 10$, $B = 11$ and $C = 12$, as in (2.3), the system (3.2) is

$$\begin{aligned} \Pi_{12P_3} &:= X_{412P_39} + X_{512P_38} + X_{612P_37} = 0 \\ \Pi_{12P_4} &:= X_{312P_4A} + X_{512P_48} + X_{612P_47} = 0 \\ \Pi_{12P_5} &:= X_{312P_5A} + X_{412P_59} + X_{612P_57} = 0 \\ \Pi_{12P_6} &:= X_{312P_6A} + X_{412P_69} + X_{512P_68} = 0. \end{aligned}$$

Finally, for the third part in the partition of $I(4, 12)$ in Lemma 1, for each set $\Sigma\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, with $1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12$, the matrix \mathcal{L}_2 of the corresponding linear equation $\Pi_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}$ of (2.3) is a matrix (row vector) with the $P_{i, \alpha_1, \alpha_2, \alpha_3, \alpha_4}$ and $P_{j, \alpha_1, \alpha_2, \alpha_3, \alpha_4}$ components equal to one and all the other components equal to zero for $1 \leq i < j \leq 6$. There are 240 such matrices \mathcal{L}_2 . The size of this vector is $1 \times C_6^{12}$. For example,

$$\Pi_{1234} = X_{512348} + X_{612347} = 0$$

with corresponding matrix $\mathcal{L}_2 := (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$, where the first 1 is in the coordinate corresponding to 123458 and the second 1 is in the coordinate corresponding to the 123467.

[illegible]
$$\mathcal{L}_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Proposition 2. *For any field \mathbb{F} , the matrix B , of size $C_4^{12} \times C_6^{12}$, associated to the homogeneous system $\Pi = \{\Pi_{\alpha_{rs}} | \alpha_{rs} \in I(4, 12)\}$ can be given by a block diagonal matrix as follows*

$$B = \mathcal{L}_4 \oplus \left(\bigoplus_{\substack{1 \leq \alpha_1 < \alpha_2 \leq 12 \\ \alpha_1 + \alpha_2 \neq 13}} \mathcal{L}_3^{(\alpha_1, \alpha_2)} \right) \oplus \left(\bigoplus_{\substack{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 12 \\ \alpha_i + \alpha_j \neq 13}} \mathcal{L}_2^{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \right),$$

$$B = \begin{pmatrix} \boxed{\mathcal{L}_4} & & & & & \\ & \boxed{\mathcal{L}_3} & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & 0 & & & \boxed{\mathcal{L}_3} & \\ & & & & & \boxed{\mathcal{L}_2} \\ & & & & & \ddots \\ & & & & & & \boxed{\mathcal{L}_2} \end{pmatrix},$$

where there are 1 matrix \mathcal{L}_4 , 60 submatrices \mathcal{L}_3 , and 240 submatrices \mathcal{L}_2 .

Proof. It follows from the observation that $I(4, 12)$ is a disjoint union of the sets described in Lemma 1 and the one-to-one relationship between those sets and their corresponding system of homogeneous linear equations. \square

For the contraction map $f : \wedge^6 E \longrightarrow \wedge^4 E$ given by (1.1), we obtain, from Proposition 2, the following consequences:

- (1) If $\text{char}(\mathbb{F}) = 3$, then $\text{rank}(B) = \text{rank}(\mathcal{L}_4) + 60 \text{rank}(\mathcal{L}_3) + 240 \text{rank}(\mathcal{L}_2) = 494$.
- (2) $\dim_{\mathbb{F}}(\ker f) = C_6^{12} - 494 = 430$.
- (3) If $\text{char}(\mathbb{F}) = 3$, then the map f is not surjective.

From Proposition 2 we calculate the codimension of the linear section $\mathbb{P}(\ker f)$ in $\mathbb{P}(\wedge^6 E)$ for any characteristic. That is

$\text{char}(\mathbb{F})$	$\text{rank}(B)$	codimension of $\mathbb{P}(\ker f)$
0	495	429
2	430	494
3	494	430
$p \geq 5$	495	429

This computation shows that the codimension of $\mathbb{P}(\ker f)$ in $\mathbb{P}(\wedge^n E)$ depends on the dimension $2n$ of the symplectic space E and the characteristic of the ground field \mathbb{F} . In a forthcoming paper the authors show that, in general, the contraction map f is surjective if and only if $\text{char}(\mathbb{F}) = 0$ or $\text{char}(\mathbb{F}) \geq m$, for a certain integer m .

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